A REFORMULATION OF BROWN REPRESENTABILITY THEOREM

SIMION BREAZ, GEORGE CIPRIAN MODOI

Abstract. A well–known result says: If a triangulated category with small coproducts satisfies Brown Representability Theorem, then every triangulated coproduct preserving functor having as domain the respective category has a right adjoint. We wonder about the converse. In the paper we provide a reformulation of Brown Representability Theorem which has some similarities with that converse.

Introduction

In the present note we outline a reformulation of Brown Representability Theorem. This result is a central piece in the theory of triangulated categories (see [2] or [4]). Consider a triangulated category $\mathcal{T}$ which has small coproducts, that is coproducts indexed over small sets. Brown Representability Theorem deals with contravariant functors $\mathcal{T} \to \mathcal{A}b$, where $\mathcal{A}b$ is the category of abelian groups, and says that such a functor is representable provided that it is cohomogical and sends coproducts into products. We consider the so called *abelianization* of $\mathcal{T}$, namely the abelian category $\operatorname{mod}(\mathcal{T})$, which satisfies the following universal property: Every cohomological functor $\mathcal{T} \to \mathcal{A}$, into an abelian category, extends uniquely to an exact functor $\operatorname{mod}(\mathcal{T}) \to \mathcal{A}$. Using this property we reformulate Brown Representability Theorem in terms of $\operatorname{mod}(\mathcal{T})$ and exact functors starting from $\operatorname{mod}(\mathcal{T})$. More precisely $\mathcal{T}$ satisfies Brown Representability Theorem if and only if every exact coproduct preserving functor $\operatorname{mod}(\mathcal{T}) \to \mathcal{A}$, where $\mathcal{A}$ is abelian AB3 and has enough injectives, has a right adjoint. In this way we obtain also some indications how far we are in order to prove (or most probably to disprove) the converse of the well–known result saying that if $\mathcal{T}$ satisfies Brown Representability Theorem then every triangulated coproduct preserving functor starting from $\mathcal{T}$ has a right adjoint. The point is the following question: Given an exact functor $\operatorname{mod}(\mathcal{T}) \to \mathcal{A}$, how can it be lifted, in a natural way, to a triangulated functor starting on $\mathcal{T}$?

For the undefined notions concerning triangulated categories we refer to [4], and for abelian categories to [1] or [5]. For general theory of categories may be also consulted [3].

2000 *Mathematics Subject Classification.* 18E15, 18E35, 18A40.
The authors were supported by the grant PN2CD-ID-489.

1. The results

Consider an additive category $\mathcal{T}$. By a *right module over* $\mathcal{T}$ (for shortly $\mathcal{T}$*-module*) we understand a contravariant functor $\mathcal{T} \to \mathcal{A}b$. Dually a *left module over* $\mathcal{T}$ (or $\mathcal{T}^{\rm op}$*-module*) is a covariant functor $\mathcal{T} \to \mathcal{A}b$. The class of all $\mathcal{T}$-module forms an abelian AB5 category, the morphisms being natural transformations, category which is denoted here by $\mathrm{Mod}(\mathcal{T})$. Note that the limits and colimits in $\mathrm{Mod}(\mathcal{T})$ are computed point-wise. Usually, this category has no small Hom-sets (another way to say this, is that $\mathrm{Mod}(\mathcal{T})$ lives in a higher universe that $\mathcal{T}$), unless $\mathcal{T}$ is essentially small (i.e. it has a small skeleton). Sometimes it is useful to restrict us to the full subcategory $\mathrm{mod}(\mathcal{T})$ of $\mathrm{Mod}(\mathcal{T})$, consisting of those $\mathcal{T}$-modules X which are *finitely presented*, that is, there is an exact sequence

$$\mathcal{T}(-,y) \to \mathcal{T}(-,z) \to X \to 0,$$

with $y, z \in \mathcal{T}$. This last category has small Hom-sets, provided that $\mathcal{T}$ does, as we may see by Yoneda lemma. The Yoneda embedding

$$\mathcal{T} \to \mathrm{Mod}(\mathcal{T}),\ x \mapsto \mathcal{T}(-,x)$$

restricts to a well defined fully faithful functor

$$H = H_{\mathcal{T}} : \mathcal{T} \to \mathrm{mod}(\mathcal{T}),\ H(x) = \mathcal{T}(-,x),$$

called also Yoneda functor, or Yoneda embedding (we will omit the index $\mathcal{T}$ if no confusion is possible). Define also $\mathrm{mop}(\mathcal{T}) = \mathrm{mod}(\mathcal{T}^{\rm op})^{\rm op}$, and denote

$$H' = H'_{\mathcal{T}} : \mathcal{T} \to \mathrm{mop}(\mathcal{T}),\ H'(x) = \mathcal{T}(x,-).$$

For the sake of clarity, from now on, we will denote by $\mathcal{T}(x,-)$ the respective (projective) object of $\mathrm{mod}(\mathcal{T}^{\rm op})$ and by $H'(x)$ the same (injective) object viewed in $\mathrm{mop}(\mathcal{T})$. It is well–known, that $\mathrm{mod}(\mathcal{T})$ (respectively $\mathrm{mop}(\mathcal{T})$) is an additive finitely cocomplete (complete) category with enough projectives (injectives), and any functor $f : \mathcal{T} \to \mathcal{A}$, into an additive finitely cocomplete (complete) category, extends uniquely, up to a natural isomorphism, to a cokernel (kernel) preserving functor

$$\hat{f} : \mathrm{mod}(\mathcal{T}) \to \mathcal{A} \quad (\check{f} : \mathrm{mop}(\mathcal{T}) \to \mathcal{A}),$$

such that $f \cong \hat{f} \circ H$ ($f \cong \check{f} \circ H'$). Note that for us, the category $\mathcal{A}$ will be always abelian. Obviously, H commutes with products, and H' commutes with coproducts which exists in $\mathcal{T}$. If, in addition, $\mathcal{T}$ has small coproducts (products) then $\mathrm{mod}(\mathcal{T})$ ($\mathrm{mop}(\mathcal{T})$) has also small coproducts (products), and the embedding H (H') commutes with coproducts (products). If this is the case, the a functor $f : \mathcal{T} \to \mathcal{A}$ preserves coproducts (products) if and only if the induced functor $\hat{f}$ (respectively $\check{f}$) does it. Finally recall that $\mathrm{mod}(\mathcal{T})$ ($\mathrm{mop}(\mathcal{T})$) is abelian, provided that $\mathcal{T}$ has weak–kernels (weak–cokernels).

From now on, the category $\mathcal{T}$ is triangulated, with small coproducts. Recall that, a functor $f : \mathcal{T} \to \mathcal{A}$, into an abelian category $\mathcal{A}$, is called *homological* (respectively *cohomological*) if it is covariant (contravariant) and sends triangles into long exact sequences. Since $\mathcal{T}$ has both weak-kernels and weak-cokernels $\mathrm{mod}(\mathcal{T})$ and $\mathrm{mop}(\mathcal{T})$ are both abelian. Moreover for a functor $f : \mathcal{T} \to \mathcal{A}$, into an abelian category $\mathcal{A}$, the induced functor $\hat{f} : \mathrm{mod}(\mathcal{T}) \to \mathcal{A}$ ($\check{f} : \mathrm{mop}(\mathcal{T}) \to \mathcal{A}$) is exact if and only if f is homological.

Recall that a functor $f : \mathcal{T} \to \mathcal{T}'$ between two triangulated categories is called *triangulated* or *exact* if it sends triangles into triangles (this implies it should commute with "shifts"). If $\mathcal{T}$ and $\mathcal{T}'$ are triangulated categories, and $f : \mathcal{T} \to \mathcal{T}'$ is an exact functor, then $H_{\mathcal{T}'} \circ f$ is homological, so it induces a unique, up to isomorphism, exact functor $f^* : \mathrm{mod}(\mathcal{T}) \to \mathrm{mod}(\mathcal{T}')$, such that $H_{\mathcal{T}'} \circ f \cong f^* \circ H_{\mathcal{T}}$. The duality functor $\mathcal{T} \to \mathcal{T}^{\mathrm{op}}$ is (contravariant) exact, so it induces as before a unique (contravariant) functor $\mathrm{mod}(\mathcal{T}) \to \mathrm{mod}(\mathcal{T}^{\mathrm{op}})$, which is not hard to see that is a duality. Therefore we obtain an equivalence of categories $E : \mathrm{mod}(\mathcal{T}) \to \mathrm{mop}(\mathcal{T})$, such that $E \circ H \cong H'$.

We call an abelian category $\mathcal{A}$ *admissible* if it is AB3 and has enough injectives. It is well–known that such a category must be also AB4.

Theorem 1.1. *The following are equivalent, for a triangulated category with arbitrary coproducts $\mathcal{T}$:*

(i) *$\mathcal{T}$ satisfies the Brown representability theorem.*

(ii) *For every homological, coproducts preserving functor $f : \mathcal{T} \to \mathcal{A}$, into an admissible abelian category $\mathcal{A}$, the induced functor*
$$\hat{f} : \mathrm{mod}(\mathcal{T}) \to \mathcal{A}$$
has a right adjoint.

(iii) *Every exact, coproducts preserving functor $F : \mathrm{mod}(\mathcal{T}) \to \mathcal{A}$, into an admissible, abelian category $\mathcal{A}$, has a right adjoint.*

(iv) *Every exact, coproducts preserving functor $F : \mathrm{mod}(\mathcal{T}) \to \mathcal{A}b^{\mathrm{op}}$ has a right adjoint.*

Proof. (i)⇒(ii). Let $f : \mathcal{T} \to \mathcal{A}$ be a homological functor into an abelian, admissible category $\mathcal{A}$. Let $I \in \mathcal{A}$ be an injective object. Then the functor
$$\mathcal{A}(f(-), I) = \mathcal{A}(-, I) \circ f : \mathcal{T} \to \mathcal{A}b$$
is cohomological, and sends coproducts into products. Then it is representable, by Brown representability theorem; so there is a unique, up to a natural isomorphism, $x_I \in \mathcal{T}$, such that $\mathcal{A}(f(-), I) \cong \mathcal{T}(-, x_I)$. Since $\mathcal{A}$ has enough injectives, the assignment $I \mapsto H(x_I)$ defines a unique, up to isomorphism, left exact functor $G : \mathcal{A} \to \mathrm{mod}(\mathcal{T})$. It is directly verified that G is the right adjoint of $\hat{f}$.

(ii)⇒(iii) is obvious, since, under the assumptions of (iii), we have $F \cong \widehat{F \circ H}$.

(iii)⇒(iv) follows immediately since $\mathcal{A}b^{\mathrm{op}}$ is admissible.

(vi)⇒(i). Let $f : \mathcal{T} \to \mathcal{A}b$ be a cohomological functor, sending coproducts into products. Then the functor
$$F : \mathrm{mod}(\mathcal{T}) \to \mathcal{A}b^{\mathrm{op}},\ F(X) = \mathrm{mod}(\mathcal{T})(X, f)$$

is exact, coproducts preserving (actually F is the composition of $\widehat{f}$ with the duality functor of $\mathcal{A}b$). By hypothesis, F has a right adjoint $G : \mathcal{A}b^{\rm op} \to \mathrm{mod}(\mathcal{T})$. We deduce $F(X) \cong \mathrm{mod}(\mathcal{T})(X, G(\mathbb{Z}))$. Further $f \cong G(\mathbb{Z})$, showing that $f \in \mathrm{mod}(\mathcal{T})$ and f has to be injective, since it represents the exact functor F. Therefore, $f \cong \mathcal{T}(-, x)$, for some $x \in \mathcal{T}$. □

We record also the dual of the preceding results (for this, we will say that the abelian category $\mathcal{A}$ is *co-admissible* if $\mathcal{A}^{\rm op}$ is admissible):

Theorem 1.2. *The following are equivalent, for a triangulated category with arbitrary products $\mathcal{T}$:*

(i) *$\mathcal{T}^{\rm op}$ satisfies the Brown representability theorem.*
(ii) *For every homological, products preserving functor $f : \mathcal{T} \to \mathcal{A}$, into a co-admissible, abelian category $\mathcal{A}$, the induced functor*
$$\check{f} : \mathrm{mop}(\mathcal{T}) \to \mathcal{A}$$
has a left adjoint.
(iii) *Every exact, products preserving functor $F : \mathrm{mop}(\mathcal{T}) \to \mathcal{A}$, into a co-admissible, abelian category $\mathcal{A}$, has a left adjoint.*
(iv) *Every exact, products preserving functor $F : \mathrm{mop}(\mathcal{T}) \to \mathcal{A}b$ has a left adjoint.*

Remark 1.3. Since, for a triangulated category $\mathcal{T}$, we have an equivalence of categories $E : \mathrm{mod}(\mathcal{T}) \to \mathrm{mop}(\mathcal{T})$, such that $E \circ H = H'$, we may freely interchange $\mathrm{mod}(\mathcal{T})$ and $\mathrm{mop}(\mathcal{T})$ in Theorems 1.1 and 1.2.

Remark 1.4. Consider a triangulated category, with arbitrary coproducts (products) $\mathcal{T}$. Theorems 1.1 and 1.2 provide reformulations of the Brown representability theorem for $\mathcal{T}$ respectively $\mathcal{T}^{\rm op}$ in terms of abelian category $\mathrm{mod}(\mathcal{T})$. Note also, Brown Representability Theorem for $\mathcal{T}$ implies every triangulated coproduct preserving functor $f : \mathcal{T} \to \mathcal{T}'$, into another triangulated category $\mathcal{T}'$ has a right adjoint. Therefore the above reformulation has some similarities with the converse of that implication, namely $\mathcal{T}$ satisfies Brown Representability Theorem if and only if every exact coproduct preserving functor $\mathrm{mod}(\mathcal{T}) \to \mathcal{A}$, into an admissible abelian category $\mathcal{A}$ has a right adjoint.

References

[1] P. Gabriel, De catégories abeliennes, Bull. Soc. Math. France, 90(1962) 323-448.
[2] H. Krause, A Brown representability theorem via coherent functors, Topology, **41**(2002), 853-561.
[3] B. Mitchell, Theory of Categories, Academic Press, New York and London, 1965.
[4] A. Neeman, Triangulated Categories, Annals of Mathematics Studies, 148, Princeton University Press, Princeton, NJ, 2001.
[5] N. Popescu, L. Popescu, Theory of Categories, Editura Academiei, Bucureşti and Sijthoff & Noordhoff International Publishers, 1979.

"Babeş-Bolyai" University, RO-400084, Cluj-Napoca, Romania
E-mail address, Simion Breaz: bodo@math.ubbcluj.ro
E-mail address, George Ciprian Modoi: cmodoi@math.ubbcluj.ro